WIRELESS NETWORKS AND THE "SMALL WORLD" PHENOMENON

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1. INTRODUCTION

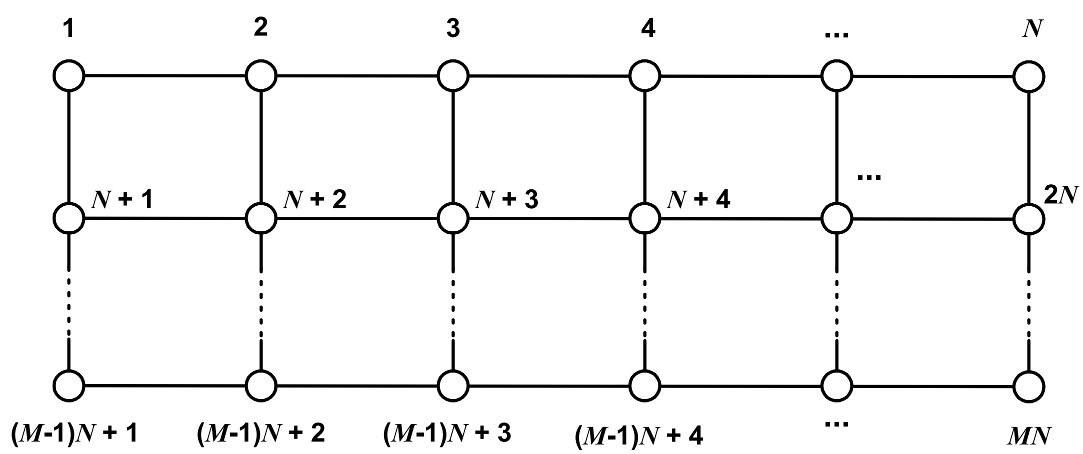
In the physics, psychology, and life science communities much attention has been given recently to the "small world" phenomenon and "small world networks" [1-3]. In this brief white paper we compare the connectivity properties of typical wireless communications networks with those of small world networks. The connectivity properties of interest are the average hop distance, also known as the characteristic path length, here denoted $\overline{m}$, and the average clustering coefficient for the neighbors of a network node, defined as the percentage of connections that exist of the $\frac{1}{2}\mathrm{k}(k-1)$ potential connections between the k neighbors of the node.

Typically, for a random network (one in which the existence of a connection between a particular pair of nodes is an independent random event), $\overline{m}$ increases linearly with the number of nodes and the clustering coefficient (C) is on the order of k/N for N nodes each having k randomly connected neighbors. However, for a small world network, $\overline{m}$ increases much more slowly with the number of nodes while having a relatively high value of C compared to a random network.

Note that for a fully connected network, $\overline{m} = C = 1$. The minimum hop distance and maximum clustering distance values are acheived simultaneously. However, for a fully connected network with N nodes, each node has $N-1$ neighbors. The networks of interest are large and they are sparse, which is to say that the degree of each node (the number of neighbors) does not grow with N.

2. MESH NETWORK PROPERTIES [4]

A mesh network of NM nodes connected by bidirectional links can be modeled as M rows of N nodes, as illustrated in the following figure:



The adjacency (one-hop connectivity) matrix for such a network, in which a 1 entry at (i, j) indicates a connection from node i to node j and a 0 entry at $(i,\ j)$ indicates no connection from node i to node j, is an $NM \times NM$ matrix with the form given by

$$A_{M\times N} = \begin{bmatrix} A_N & I_N & 0 & \cdots & 0 \\ I_N & A_N & I_N & \cdots & 0 \\ 0 & I_N & A_N & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & A_N \end{bmatrix} \tag{1}$$

where I_N is the $N \times N$ identity matrix and A_N is the $N \times N$ adjacency matrix for a single row of N nodes connected in tandem. The structure of A_N is easily determined to be a matrix of 0s, except for $N-1$ 1s on the first upper diagonal and $N-1$ 1s on the first lower diagonal, for example,

$$A_6 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \tag{2}$$

The one-hop connectivity of the network, defined as the fraction of the $NM(NM-1)$ possible links that are operative, is simply the sum of the elements of $A_{M\times N}$ divided by $NM(NM-1)$, or

$$\text{Connectivity} = \frac{M \times 2(N-1) + 2(M-1) \times N}{NM(NM-1)} = \frac{2[2NM - M - N]}{NM(NM-1)} \tag{3}$$

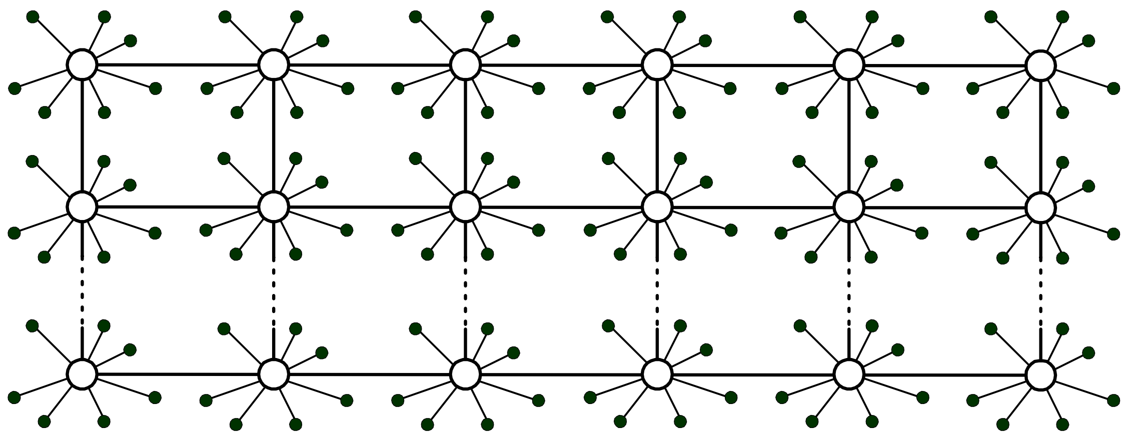
In addition to one-hop connectivity, we are interested in the hop distance between each pair of nodes, defined as the minimum number of links needed to be traversed in order to connect the pair. The average hop distance for an $M \times N$-node mesh network is

$$\overline{m} = \frac{N+M}{3} \tag{4}$$

By inspection of the network diagram or the adjacency matrix for a mesh network, we find that the nodes have two, three, or four neighbors and that none of any node's neighbors are neighbors of each other. Therefore, the clustering coefficient for the mesh network equals zero.

3. TELEPHONE NETWORK: MESH WITH LOCAL STARS

The physical connectivity of a wireless or cellular telephone network resembles that of a mesh (representing the public switched telephone network (PSTN)) in which mesh node is an access node, and many customer nodes connected to each mesh node but not to any other node, as illustrated in the next figure. (We ignore local switches for this discussion.)



For an $M \times N$-node mesh "backbone" network with K "customer nodes" per backbone node, the one-hop connectivity is increased over that for the mesh network alone by the addition of the many one-hop links between backbone and customer nodes:

$$\text{Connectivity} = \frac{2[2NM - M - N] + 2NMK}{NM(K+1)(NM(K+1)-1)} \tag{5}$$

The average multihop distance is affected by the addition of the many one-hop links, but for large K tends toward the $\overline{m}$ for the mesh network, plus two:

$$\begin{aligned}\overline{m} = \frac{1}{NM(K+1)(NM(K+1)-1)} \Bigg[& NM(NM-1) \cdot \frac{N+M}{3} + 2NMK \cdot 1 \\ & + \Big[NMK(K-1) \cdot 2\Big] + 2NMK \cdot (NM-1)\left(\frac{N+M}{3} + 1\right) \\ & + NMK(NM-1)K\left(\frac{N+M}{3} + 2\right)\Bigg] \qquad (6) \\ \approx \frac{1}{K^2} \cdot \frac{N+M}{3} + \frac{2}{NMK} + \frac{2}{NM} + \frac{2}{K}\left(\frac{N+M}{3} + 1\right) + \left(\frac{N+M}{3} + 2\right) & \qquad (7)\end{aligned}$$

where the approximation in (7) is for large values of N, M, and K. By inspection, the clustering coefficient equals zero because none of the customer nodes are (1-hop) neighbors of each other.

4. MESH WITH LANS

If the K customer nodes for a particular backbone node, together with the backbone node, are configured as a fully connected wired or wireless LAN, then the clustering properties of the overall network are affected significantly but the multihop connectivity is virtually the same—the term in bold brackets in (6) becomes $NMK(K-1)\cdot 1$.

The clustering coefficient for each customer node in a LAN equals 1. The clustering coefficient for each backbone node depends slightly on the position of the node in the mesh network. Some backbone nodes, for example, have four backbone node neighbors and K customer neighbors. The average cluster coefficient for all the nodes is

$$\begin{aligned} C = \frac{1}{NM(K+1)}\Big[& NMK\cdot 1 + 4\cdot\frac{K(K-1)}{(K+2)(K+1)} \\ & + [2(N-2)+2(M-2)]\cdot\frac{K(K-1)}{(K+3)(K+2)} \\ & + (N-2)(M-2)\cdot\frac{K(K-1)}{(K+4)(K+3)}\Big] \end{aligned} \tag{8}$$

Note that

$$C \geq \frac{K}{K+1} \tag{9}$$

which gives a very high value of the clustering coefficient compared to those cited for example small world networks.

REFERENCES

[1] D. J. Watts and S. H. Strogatz, "Collective Dyanmics of 'Small-World' Networks," *Nature*, Vol. 393, pp. 440-442, 4 June 1998.

[2] M. E. J. Newman, "Models of the Small World: A Review," available as cond-mat/0001118 v2 9 May 2000.

[3] Anon., "Kevin Bacon, the Small-World, and Why It All Matters," Santa Fe Institute *Bulletin*, Vol. 14 No. 2, Fall 1999, available at http://www.santafe.edu/sfi/publications/Bulletins/bulletinFall99/workInProgress/smallWorld.html

[4] L. E. Miller, "Connectivity Properties of Mesh and Ring/Mesh Networks," April 2001, white paper available at http://w3.antd.nist.gov/wctg/netanal/Meshcon.pdf